THE EDPLUS ALGORITHM

This document describes some of the technical aspects of the Edplus algorithm. After a brief overview, the main principles underlying the algorithm are discussed in turn, followed by a comparative discussion of its performance relative to alternative algorithms, including spaced repetition.

1. Technical summary

What it is: The Edplus algorithm is a general-use adaptive algorithmic framework for intelligent computer-generation of multiple choice questions to assist with learning facts in an optimised manner.

Working examples: Number facts (e.g., times tables for primary school children), phonics: 100's of items. Foreign language vocabulary and grammar, writing systems (e.g., Chinese characters), science facts: 1000's of items.

The main features.

(1) (Topology of knowledge). A key concept underlying the Edplus algorithm is the idea that the set of knowledge to be acquired has a geometric structure. It formalises the intuitive notion that the likelihood of learning a new piece of information depends on what one already knows. For example, if a person knows the word for 'train' in a language, then they are more likely to be interested in, and hence memorize, the related word 'ticket' than another randomly-chosen word. Likewise, the operation '$3 + 4$' is closely related to '$4 + 13$', but unrelated to '7×11'. The concept of 'nearby' or related knowledge is encoded mathematically by a variant of the geometric notion of topology on the set of knowledge that needs to be acquired.

(2) By exploiting this topological structure, the Edplus algorithm ascertains the user's knowledge map as rapidly as possible and preferentially asks questions in the neighbourhood of their knowledge hotspots. Over time, these pockets of mastered facts will grow like crystals until they fill the whole knowledge space (see figure 4). The topological structure of knowledge sets are optimised from user population data.

(3) The Edplus algorithm combines the topology of knowledge with a spaced-repetition system per question. The result is a dynamic probabilistic model which selects questions for memorisation according to a varying probability distribution. This is updated in real time with every

right or wrong answer. Since it is not rule-based, it behaves differently for each user, evolving as their knowledge state changes. As a consequence, the behaviour of the algorithm does not impose any particular order on learning. If the user starts getting answers right on a new topic then the algorithm will, by its very nature, encourage it. This is in the spirit of interest-driven or ability-driven learning.

Further comments. The method of delivery of the multiple-choice questions plays an integral part in the Edplus algorithm design.

- *Continuous learning*. The algorithm actively teaches the user the correct answer to a question even if it is answered incorrectly (for which there is no penalty). In this way, the learning of new facts occurs continuously as an integral part of testing.
- *Mastery*. The algorithm emphasises mastery. Concepts which have been demonstrably mastered by the user are asked less frequently, increasing efficiency of memorisation and avoiding over-learning.
- *Robustness*. User data is inherently noisy, especially in the case of children who are easily distracted and can answer questions inconsistently depending on their mood. The algorithm gathers information about the user's knowledge state in a robust manner which irons out the random fluctuations associated with lucky guesses or clumsy finger errors.
- *Hints to reinforce concepts and learning strategies*. The topology of knowledge is underpinned by logical concepts and organising principles. For example, the law of commutativity of multiplication explains why $a \times b$ and $b \times a$ are related to each other, and hence nearby in the topology of the knowledge space. In the case of languages, grammatical concepts explain how word derivatives such as 'eat', 'eats', 'eating' and 'eaten' are related. Although the algorithm focuses primarily on rote learning, conceptual understanding of general principles can be reinforced by hints, which are selected by the algorithm in tandem with the questions, and are generated as a function of the user's knowledge state and response history. For example, a user who has demonstrated a good understanding of the number fact $2 \times 5 = 10$, may be asked the related question 'what is 10 divided by 2'? A hint explaining how the two questions are related can reinforce the notion that division is the inverse operation of multiplication.

2. Underling principles and assumptions

The Edplus algorithm is based on three principles:

(1) *Practice makes perfect*. Mastery is attained by frequently engaging with material of appropriate difficulty. Furthermore, regular exposure promotes familiarity, and in turn leads to increased confidence.

(2) *We are more likely to learn facts relevant to what we already know and understand*. By exploiting the interrelations between facts, the Edplus algorithm preferentially selects questions that are in the neighbourhood of the user's existing knowledge points and relevant to their centres of interest in order to maximise uptake of new information.

(3) *Rehearsal forges long-term memories*. The Edplus algorithm uses established results in the science of memory formation and forgetting, to optimise when to test the user on recently learned facts in order to consolidate long-term retention.

Point (1) is more or less self-evident. We first discuss the well-known principles underlying (3) which form the basis for many spaced repetition systems used in language learning and education platforms. The main novelty is to combine this, in an algorithmic framework, with a concept of correlated knowledge (2), which will be discussed immediately afterwards.

3. The science of forgetting

In the late 19th century, Ebbinghaus performed the following experiment. He would memorise a certain number of items and test himself after fixed time intervals to see how many he could remember. He discovered that he would forget items at an exponential rate, leading to what is now called the Ebbinghaus forgetting curve, which has been replicated in numerous experiments [3]. It states that the quantity of information retained after each time interval decreases by a fixed percentage until it is reinforced by revising, or by testing. For example, if we learn 100 items today, we may remember only 50 of them tomorrow, 25 the day after, and so on. Immediately after testing these items of information, the number that we remember could revert to 100 (they are fresh in our mind), but thereafter will steadily be forgotten again. But this time the rate of forgetting may have improved: it may be 100 today, 80 tomorrow, 64 the day after, and so on. This model is the basis for the method of spaced repetition which posits that learning is optimised if items are tested at around the point in time that they are about to be forgotten [4, 5].

A spaced repetition model tests questions at ever-increasing intervals based on whether they are answered correctly or incorrectly. It can be highly effective and avoids 'overlearning': spending too much time testing items of knowledge which have already been mastered. Spaced repetition is widely used in language-learning platforms and extremely efficient compared to random testing, which tests in a redundant manner.

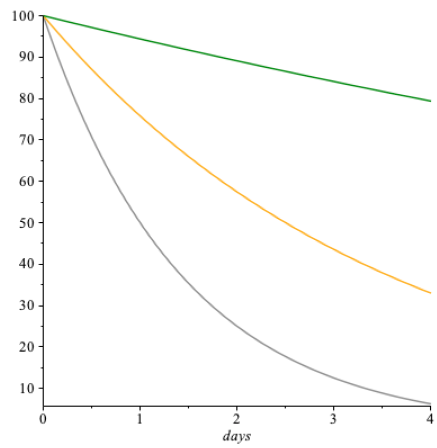

FIGURE 1. Three idealised forgetting curves representing how much information is retained after a number of days. The rate of forgetting for the green curve is slower than for the grey curve: it represents material which has is more consolidated. See [3] for real-life images of forgetting curves.

3.1. **Weakness of spaced repetition systems.** A fundamental deficiency of a simple spaced-repetition model is that it assumes that items of knowledge are completely independent: i.e., each item is equally likely to be memorised. This completely ignores organisational priniciples and patterns which are key to a deeper conceptual understanding of the subject-matter. So although a spaced repetition system is highly effective at testing a *specified* set of items which need to be learned, it is incapable of deciding *which* items should be made available for study in the first place.

In practice, this often requires human intervention, where the user has to select sets of flashcards for revision, or is decided for the user in a prescriptive manner where the next choice of topic has been determined in advance. Both approaches are unsatisfactory and are solved by the Edplus algorithm using the topology of knowledge.

3.2. **Prescriptive versus interest-driven learning.** Traditionally, topics for study are prescribed by a teacher, and students learn topics in a fixed order. In the case of elementary arithmetic, this could be by first learning the 'easier' times tables: 2, 5 and 10 before then moving onto the rest.[1] In the case

[1]Opinions differ on what qualifies as 'easy': the national curriculum has changed policy on times tables, and approaches vary from country to country.

of learning languages, classroom textbooks and language learning applications move from topic to topic in a predetermined way (first 'transport', then 'the office', 'holidays', etc). It stands to reason that this cannot be the most efficient approach to learning as it takes no account of individual abilities and preferences. For example, a language-learner may find vocabulary hard to recall if they perceive that it is irrelevant to their life.

A truly adaptive computer learning system must take into account individual abilities when offering topics for study. Interest-driven learning is where the topics for study are tailored to the knowledge state of the user. The Edplus algorithm can identify the strengths and interests of the user based on their previous answers and will select the next questions for study based on what is most relevant to the user's current knowledge state, and based on the experience of other similar users.

4. Topology of knowledge

4.1. **Correlations between items of knowledge.** Correlated knowledge expresses the idea that certain facts are more closely related to each other than others. This is particularly obvious in the case of mathematics.

For example, the operations '$2+5$' and '$5+2$' are closely connected because they are related by the rule of commutativity of addition: '$a + b = b + a$'. If a child consistently answers one of these two questions correctly but not the other, then it suggests that they may struggle with the concept of commutativity in general. These two calculations are in turn closely related to the operations '$7 - 2$' and '$7 - 5$' since subtraction is the inverse of addition. Indeed, all mathematical concepts and theorems emerge from the patterns and correlations between mathematical facts. They are by definition the organisational principles for these facts. This is one reason why familiarity with facts is a crucial part of mathematical understanding.

The times tables are a familiar example of a highly correlated data set which contains many patterns. For example, the 10 times table are all related to each other because they are generated by the simple rule 'append a zero'. Some of these correlations in simple cases are illustrated in Figure 2, which may give an idea of the potential complexity of this knowledge set.

The notion of correlated knowledge is also familiar in the somewhat different context of language. For example, the words for 'bus' and 'ticket' are conceptually related and often occur in conjunction with each other. A user familiar with the word for 'beach' is more likely to be interested in the word for 'sea' than one who is not. Other examples of correlated words are grammatical and related to the concept of morphemes: the words 'agree', 'disagree' and 'agreement' are closely related since they have a common root [7]. Correlated words often underlie an etymological or grammatical principle (the negation 'dis-'), a theme (words related to 'travel', 'work' etc), a contextual relation

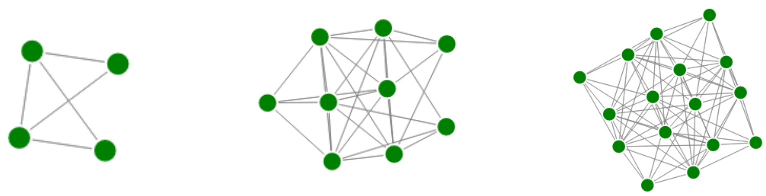

FIGURE 2. Correlations in the times tables up to two (left), up to three (middle), and up to four (right). The graph on the left has four nodes corresponding to the operations 1×1, 1×2, 2×1, and 2×2. Each link represents a logical relation between operations (for example, 1×1 and 2×2 are not *a priori* related to each other, but 1×1 and 1×2 are related via the concept of 'multiplication by 1'). The graph in the middle has nine nodes (1×1, 1×2, 1×3, $\ldots$, 3×3); the graph on the right sixteen. The full 12×12 times table is too complex to represent clearly in a two-dimensional figure.

(e.g., 'glass' and 'water'), or it could simply be the case that words are statistically correlated within the population knowledge state (e.g., 'brown' and 'cow', or 'watched' and 'pot').

In conclusion, most sets of facts to be learned carry more structure than a set of independent data points, and are subject to organisational principles. The latter are discovered and assimilated through familiarity with the examples, and are the key to unlocking the wider structure of the knowledge set. This is especially true of mathematics, where number facts are highly interrelated and reveal many patterns.[2] It stands to reason that the best way to acquire this knowledge is via making use of this structure.

4.2. **Topology of knowledge and the Edplus algorithm.** The Edplus algorithm exploits the notion of correlated knowledge to choose questions which are related to the users' existing body of knowledge. The idea that some facts are closer than others is formalised mathematically by a concept related to the notion of topological space, where the set of knowledge items is endowed with a loose but potentially very complex geometric structure. Its exact form is determined by conceptual principles and supported by population data to ascertain relationships between knowledge points: the shape or topology of a single knowledge space simultaneously encapsulates the learning patterns of typically tens or even hundreds of different users. Combined with Ebbinghaus'

[2]Indeed many of these patterns underlie fundamental concepts in arithmetic. Of particular importance in education are: commutativity of multiplication, subtraction as the inverse of addition, division as the inverse of multiplication, and the distributivity of addition and multiplication, according to [6].

forgetting curve model we have derived an evolving probabilistic dynamical system on the knowledge space which behaves in a unique manner for each individual user. It makes no assumptions about which order best to learn. Topics which are nearby (with respect to the geometric structure or topology of knowledge) to the user's existing knowledge hotspots are preferentially selected over those which are far away.

5. Comparisons

We now discuss the efficacy of the Edplus algorithm as compared to a simple spaced repetition model.

5.1. **Knowledge state.** Figure 3 illustrates the knowledge state of a typical 8 year old user, based on gameplay on Seal Saver over a period of several weeks. Since every question in the times table has been asked several times, we can expect the figure below to be a fairly accurate representation of the user's knowledge state at that point in time.

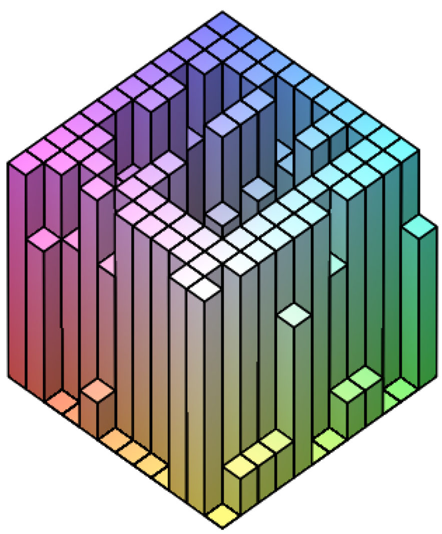

FIGURE 3. The times table knowledge state of an 8 year old user, as represented within the Edplus algorithm. Each square represents a times table with 1×1 in the top corner, and 12×12 in the bottom corner: the higher the box, the greater the level of understanding. The data is collated in a robust way from the user response history which can be inconsistent (children are easily distracted) and constantly changing.

5.2. **Comparative tests.** An automated robot based on the above real-life user was pitched against different testing algorithms for comparison. Every time the robot is tested on a question it answers it correctly if the original user had mastered the question, but picks an answer at random otherwise[3]. The robot was then tested over several hundred questions against the Edplus algorithm on the one hand, and against a simple spaced repetition system on the other.

This entire experiment can then be repeated any number of times, but the results are highly consistent. To make the comparison completely fair, the parameters for the spaced repetition system were set identically in both cases (since Edplus contains, *per question*, a spaced repetition system).

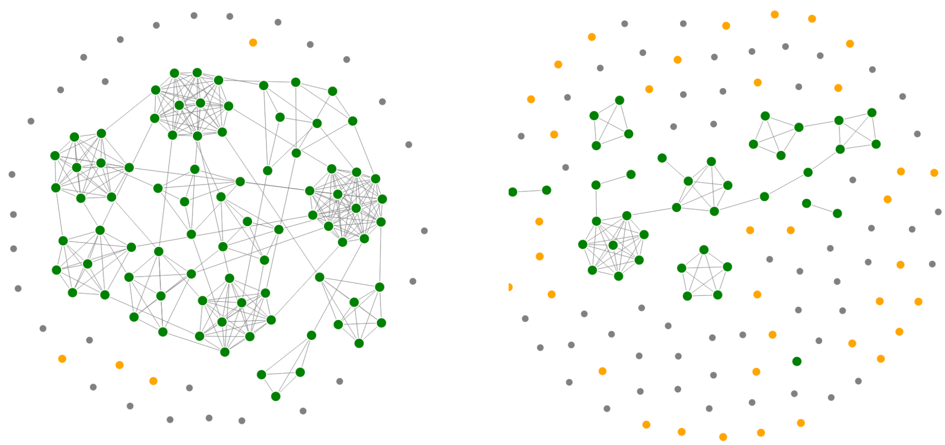

FIGURE 4. The result of a robot run against the Edplus algorithm (left) and spaced repetition system (right). The robot is asked times tables questions 600 times. Each node represents a different question which has been asked at least once. It is green ('mastered') if it has been answered correctly the last 3 or more times in a row, orange ('learning') if it has been answered correctly the last 2 times in a row, and grey ('started') otherwise. The number of green nodes attained by the Edplus algorithm is about double those attained by spaced repetition (82, compared to only 40). In addition, the Edplus algorithm is more focused, as it has significantly fewer grey nodes.

The results of one such run are depicted in Figure 4. This is a screen shot of an animation which can be viewed at

`https://Edplus.app/comparison/.`

[3]in fact, it answers it correctly to a high probability proportional to how well the original question was mastered. This more accurately reflects occasional user errors.

Typically, after 600 questions, the robot attains mastery in double the number of questions when using the Edplus algorithm compared to a simple spaced repetition system. Furthermore, the testing with Edplus is more focused, as it asks fewer repeatedly-failed or inappropriate questions.

The links between the nodes in Figure 4 represent some, but not all, of the correlations that the Edplus algorithm uses to determine the structure underlying the robot's knowledge state.

Figure 5 represents the number of questions mastered over time for a run of the robot over a similar timeframe.

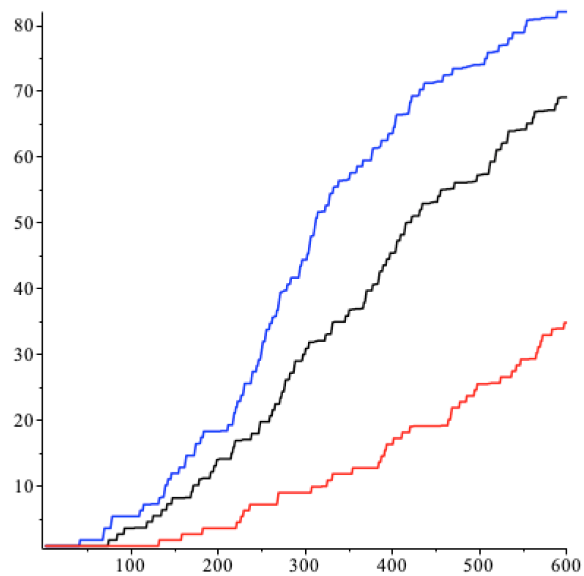

FIGURE 5. Comparison between three algorithms: Edplus (blue), random (black) and spaced repetition (red). The data was generated via a robot answering questions according the knowledge state of a real-world user against the three algorithms. The horizontal axis denotes the number of questions asked, the vertical axis how many of those questions have been 'mastered' (answered correctly at least three times in a row). For example, Edplus had to ask the robot roughly 300 questions to certify mastery of 40 knowledge points. Spaced repetition took over 600 questions to achieve this.

The effectivity of an artificial intelligence tutoring system necessitates an understanding the users' knowledge state: after all, a machine cannot effectively tutor until it knows the user's ability and background. In Figure 5 we compare the speed of acquisition of the knowledge state of a real user for the Edplus algorithm versus a random and spaced repetition system. We see that the Edplus algorithm outstrips spaced repetition by well over a factor of two. Perhaps paradoxically, spaced repetition actually performs worse than random in terms of probing the knowledge state. This is because it supresses questions answered correctly and hence takes longer to certify mastery. However, in the long run, spaced repetition is massively more efficient than random at imparting information. The Edplus algorithm combines the best of both worlds: it has the fastest knowledge acquisition, but also matches the efficiency of a spaced repetition system in terms of the quantity of new information that it teaches.

6. Facts versus concepts

It is sometimes argued that learning facts is old-fashioned and that we should teach concepts instead. Concepts are vitally important, and that is why the teacher and classroom will always play a crucial role in education. But true mastery also requires individual practice, and this can effectively be done outside the school environment.

One of our immediate goals at Edplus is to take the rote learning, repetition and memorisation out of the classroom, freeing up the teacher to get on with the task of promoting understanding. For example, in the United Kingdom, four years of the national curriculum [1] are devoted to learning times tables. Compulsory testing [2] will be introduced for all year 4 students starting from 2020. The times tables contain just 72 number facts which are highly interrelated and contain many patterns which help to understand their structure[4]. We believe that fluency in times tables and confidence in the basic operations of arithmetic can be achieved at home. In fact, any facts that must be learned as part of the curriculum can be studied outside school: language vocabulary, spelling, capital cities, chemical formulae, and so on. Mastery of any discipline requires a student to internalise a certain number of facts and have them at their fingertips. This cannot be done on their behalf.

The process of learning to drive a car is a familiar example, possibly because it tends to occur in adulthood and we can more easily remember a time when we didn't know how to drive. Although the principles of driving are perfectly simple: turn the wheel to steer, brake and accelerate with the pedals; it still takes many hours of practice to become a proficient driver. It is only after these skills have become deeply ingrained and second nature that the driver's mind is free to think about the other important tasks such as safely navigating through traffic, or finding their way to their destination. Mathematics is no different, and like any other skill, requires practice. Only after mastering the rules of arithmetic and acquiring a deep familiarity with them can one progress

[4]Some people argue that learning times tables is useless since they can be derived from first principles. However, our understanding of principles often reposes on the fact that we learned times tables as children. To illustrate the point, the reader might like to answer the following question from first principles and without recall of times tables: bookie A offers you odds of 5:8 on a bet, and bookie B odds of 7:11. Which one should you take?

to more advanced tasks, such as fractions, or simultaneous equations. Like the scaffolding used to build a house, mental arithmetic can be cast aside once the foundations of mathematical understanding are solid. Failure to internalise the rules of arithmetic makes it increasingly difficulty to progress to more sophisticated concepts. For all these reasons, Edplus will make the times tables component of its app freely available to the public.[5]

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[5]Figures 1,3,5 were generated using Maple™. Figures 2,4,6 were generated using D3.js